Abstract

The dissertation is dedicated to the analysis of light propagation in curved spacetimes of general relativity, within the framework of geometrical optics approximation. Its main idea is to apply effective methods of dynamical systems theory to this classical problem, inherently linked with the main themes of general relativity.

The starting point is the equivalence principle, which implies that the trajectories of light rays in spacetime are null geodesics. The equations describing geodesics take the form of second-order Lagrange equations derived from a simple Lagrangian quadratic in velocities, provided the evolution parameter (the so-called affine parameter) is appropriately chosen. Null geodesics are additionally characterized by the vanishing of the "energy"integral. In flat spacetime, Fermat's principle of the shortest time is a useful tool for describing the propagation of light rays. Its generalization to curved spacetimes describing constant gravitational fields, both static and stationary, has long been established. Fermat's principle has also been formulated for arbitrary gravitational fields; however, in this case, it has a less natural and functional form. One of the main results of the dissertation is a new version of Fermat's principle for arbitrary gravitational fields. The starting point is a simple theorem proven in the dissertation. We consider a Lagrange function L that satisfies the following conditions: (i) L is a non-degenerate, homogeneous function of generalized velocities of the second degree; (ii) L does not depend on the evolution parameter. By selecting an arbitrary value of energy, one can solve the equation $E = L$ for one of the generalized velocities. It turns out that the obtained solution defines a new Lagrange function describing reduced trajectories in the configuration space parameterized by the remaining generalized coordinates. This simple and elegant result comes at a cost: $\mathcal{L}$ is a generalized Lagrangian that depends on the action variable. Such generalization is described by the so-called Herglotz formalism, which is as well-developed as the standard formalism of analytical mechanics. Consequently, the version of Fermat's principle proposed in the dissertation provides effective tools for analyzing various specific problems, including an alternative formulation of Fermat's principle, which finds clear justification within the framework of the Herglotz variational principle.

The second main result of the dissertation is based on the observation that in the case of geometries describing black holes, the equations derived from Fermat's principle describe nonlinear oscillations. In particular, in the asymptotic region, these are small oscillations. Approximate solutions to equations describing small nonlinear oscillations take the form of a series expansion in the amplitude of such oscillations. The correct expansion is described by the Lindstedt–Poincaré algorithm. Applying this algorithm to the problem of light propagation in the asymptotic region leads to a description that accurately reflects both the qualitative and quantitative characteristics of the trajectories. In particular, it provides an elegant method for calculating the deflection angle of light rays in the field of a black hole.

The last problem considered in the dissertation concerns the propagation of light rays in the Kerr metric. From the perspective of the dynamics describing geodesic trajectories in four-dimensional spacetime, this is an integrable system in the sense of Arnold–Liouville. The integrals of motion are: the Lagrangian L (coinciding with the Hamiltonian), the generalized momenta conjugate to the temporal and azimuthal coordinates, and the so-called Carter constant, generating symmetry transformations that are not point transformations. In the dissertation, null geodesics are analyzed within the framework of Fermat's principle. An appropriate Lagrange function can be chosen so that the evolution parameter is the azimuthal angle. This yields a two-dimensional dynamical system with a configuration space parameterized by the radial variable r and the zenith distance

θ. The corresponding Hamiltonian is quite complex, but the problem can be simplified by applying the so-called constant-coupling metamorphosis method. The unparametrized trajectories of the original Hamiltonian, corresponding to different energy values, can be described as zero-energy trajectories of a family of Hamiltonians which parameters are functions of the original energies. This results in a much simpler description of null geodesics in the Kerr metric within the framework of a two-dimensional integrable system in the sense of Arnold–Liouville. An additional redefinition of the evolution parameter allows complete separation of variables, reducing the problem to the dynamics of two independent nonlinear Newtonian oscillators.

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